FROM LOCAL TO GLOBAL EQUILIBRIUM STATES: THERMODYNAMIC FORMALISM VIA AN INDUCING SCHEME

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ABSTRACT. We present a method to construct equilibrium states via inducing. This method can be used for some non-uniformly hyperbolic dynamical systems and for non-Hölder continuous potentials. It allows to prove the occurrence of phase transition.

1. SETTINGS

1.1. **Goal.** We consider a dynamical system (X, f), where X is a compact metric space and f is topologically mixing and local homeomorphism. For x in X, f_x^{-1} is the inverse branch defined by x. It is a homeomorphism defined on a neighborhood of $f(x)$ onto its image which is a neighborhood of x.

If $\phi : X \to \mathbb{R}$ is a continuous function or at least a Borel function, the pressure of ϕ is

$$\mathcal{P} := \sup \left\{ h_\mu(f) + \int \phi \, d\mu \right\},$$

where $h_\mu(f)$ is the Kolmogorov entropy. The supremum is taken over the set of invariant probabilities. A measure which realizes the supremum is called an *equilibrium state for* ϕ. In the following it will be also referred to as a *global equilibrium state*. In the following, *studying the thermodynamic formalism* means study existence, uniqueness and other properties of global equilibrium states.

It is well-known that if (X, f) is uniformly hyperbolic and ϕ is Hölder continuous, then there exists a unique equilibrium state (see *e.g.* [2, 7]). In that case it is also a *Gibbs state* and is fully supported. The main heuristic explanation for this result is that hyperbolicity and Hölder regularity combine themselves and allow to construct the equilibrium state via the spectral elements of the *transfer operator*.

The existence of equilibrium states for less regular potentials and/or for systems with weaker hyperbolicity properties is a challenging task. In that case, the study of the spectral properties of the transfer operator is usually much harder. It turns out that, for several cases, one strategy is to consider inducing scheme.

Inducing scheme may also be a solution to deal with another question, related to the notion of *phase transition*. Beyond the question of existence of some equilibrium state, which can follows from abstract properties, one may want to get information on that equilibrium state as *e.g.* if it is (or not) fully supported. For this kind of question, inducing scheme can be a solution; it is actually a way to prove that an equilibrium state gives or does not give positive weight to some special set (see *e.g.* [3]).

Date: Version of December 22, 2013.
AMSC : 37D35
Author wants to thank HB, SP and JRC for reading and advise.

The goal of this note is to present a method based on inducing scheme to answer to these two previous questions. This method was actually presented in [9] for the case of uniformly hyperbolic dynamical system and Hölder continuous potential. It was then developed and extended in further works of the author; the note summarizes here all the principal steps using more general settings. Such a more general presentation allows the use of the method for more general settings that the initial ones.

1.2. **Markov set and induced system.** We assume there exists some Markov set R: this is a proper set $R = \overline{\mathring{R}}$ such that for every n and for every $x \in \mathring{R} \cap f^{-n}(\mathring{R})$, the set $C_{n,f}(x) := f_x^{-n}(R)$ is contained in R and $f^n(C_{n,f}(x)) = R$.

We consider the induced subsystems (R, F) where F is the first return map:

$$F(x) = f^{\tau(x)}(x) \ \text{ if } \tau(x) := \min\{k > 0, f^k(x) \in R\}.$$

The map F is not necessarily well-defined everywhere and may also be multivalued. However it is well-defined μ-a.e. for any invariant probability μ satisfying $\mu(R) > 0$ (Poincaré's recurrence theorem). Moreover, the main point is that the Markov property allows to well define the inverse branches: if x belongs to $\mathring{R}$ and $F(x)$ is well defined and also belongs to $\mathring{R}$, then we set $C_{1,F}(x) := C_{\tau(x),f}(x)$. It is called the *1-cylinder of x (with respect to F)*. The integer $\tau(x)$ is called the *return-time* of the 1-cylinder. By construction $C_{1,F}(x)$ is a proper set, and the intersection of two different 1-cylinders has empty interior. They also form a *partition* of R,

- up to points which never return to R by iterations of F,
- and up to the fact that two 1-cylinders may have non-empty intersection (on their borders).

For x in the interior of a 1-cylinder C, $\tau(x)$ is well-defined and coincides with the return-time of the cylinder; this may not hold for point in ∂C. However we set $F(x) = f^n(x)$ if x belongs to ∂C and the return time for C is n. We point out that F may thus be multi-valued on these points, but these values agree with the inverse branches and this is what is important for the method we present here. Then, the Markov property yields that for each x in R and for each 1-cylinder C, there exists a unique $x' \in C$ such that $F(x') = x$. The set of x''s is denoted as $Pre(x)$.

The main question we are interested in is :

Question 1. Is there a way to study the Thermodynamic formalism for (R, F) and to recover (or to study the properties of) the equilibrium state for (X, f) and ϕ ?

The method we present here gives a positive answer to that question, up to some reasonable assumptions on ϕ. Reasonable means for instance, that it is possible to study the thermodynamic formalism for (R, F).

1.3. **Hypothesis on ϕ. A more precise question.** Consider some f-invariant probability measure $\widehat{\mu}$. We assume $\widehat{\mu}(R) > 0$. Then, the conditional measure $\mu := \dfrac{\widehat{\mu}(. \cap R)}{\widehat{\mu}(R)}$ is F-invariant. We remind the notation

$$S_n(\phi) := \phi + \phi \circ T + \ldots + \phi \circ T^{n-1}.$$

In particular $S_{\tau(\cdot)}(\phi)(\cdot)$ maps x to $S_{\tau(x)}(\phi)(x)$.

By the Abramov formula (see [13] p. 257-258) we get

$$\begin{aligned}
h_{\widehat{\mu}}(f)+\int\phi\,d\widehat{\mu} &\leq \mathcal{P} \text{ with equality iff } \widehat{\mu}=\text{equil. state}\\
&\Updownarrow\\
h_{\widehat{\mu}}(f)+\int\phi\,d\widehat{\mu}-\mathcal{P} &\leq 0 \text{ with equality iff } \widehat{\mu}=\text{equil. state}\\
&\Updownarrow\\
\widehat{\mu}(R)\left(h_{\mu}(F)+\int S_{\tau(.)}(\phi)-\mathcal{P}.\tau(.)\,d\mu\right) &\leq 0 \text{ with equality iff } \widehat{\mu}=\text{equil. state}\\
&\Updownarrow\\
h_{\mu}(F)+\int S_{\tau(.)}(\phi)-\mathcal{P}.\tau(.)\,d\mu &\leq 0 \text{ with equality iff } \widehat{\mu}=\text{equil. state}
\end{aligned}$$

This simple sequence of inequalities shows that the thermodynamic formalism for (X,f) and ϕ is related to the thermodynamic formalism for (R,F) and $S_{\tau(.)}(\phi)(.)$.

For x in $C_{1,F}(y)=C_{n,f}(y)\subset F$, we set $\Phi(x):=S_n(\phi)(x)$. Our main assumptions on ϕ are that it is possible to study the thermodynamic formalism for Φ. Hypotheses are listed along the way. We first assume:

(H1) Φ is continuous on each 1-cylinder.
(H2) There exists C such that for x and y in the same 1-cylinder, $|\Phi(x)-\Phi(y)|\leq C$.

For $Z\in\mathbb{R}$ we set,

$$\mathcal{L}_Z(\psi)(x):=\sum_{y\in Pre(x)}e^{\Phi(y)-Z.\tau(y)}\psi(y). \tag{1}$$

This is the transfer operator for (R,F) and for the potential $\Phi-Z.\tau(.)$.

Question 2. For which Z can we study the thermodynamic formalism ?

Definition 1.1. *Any equilibrium state for (R,F) and for $\Phi-Z.\tau(.)$ is called a local equilibrium state (associated to the parameter Z). It will be denoted by μ_Z (if it exists).*

Question 3. Among the measures μ_Z, can we recover an equilibrium state for (X,f) and ϕ ?

Roughly speaking, Question 3 means that we want to find Z such that the local equilibrium state say μ_Z for $\Phi-Z.\tau(.)$ is the induced measure of a global equilibrium state (for (X,f) and ϕ). It is a reformulation of Question 1.

2. ANSWERS TO QUESTIONS

2.1. Local thermodynamic formalism.

Proposition 2.1. *There exists a critical $Z_c\geq-\infty$ such that*
$-$for every $Z<Z_c$ and for every $x\in R$, $\mathcal{L}_Z(\mathbb{1}_R)(x)=+\infty$,
$-$and for every $Z>Z_c$, for every $\psi:R\to\mathbb{R}$ continuous and for every $x\in R$, $\mathcal{L}_Z(\psi)(x)$ converges[1].

Proposition 2.2. *If ϕ is continuous, then $Z_c\leq\mathcal{P}$.*

[1]In the following one will thus simply mention "$\mathcal{L}_Z(\mathbb{1}_R)$ converges".

Proof. Actually, Z_c can be realized as the pressure for some measure which does not give positive weight to R. We refer to Prop. 3.10 in [12] for a proof with a possible discontinuous potential. □

In the case of subshift of finite type, we can either get a better characterization for Z_c.

Theorem 1 (see [4], Lem. 3.4)**.** *If (X, f) is a mixing subshift of finite type and R is a cylinder and ϕ is continuous, then Z_c is the pressure for ϕ of the set of points whose trajectory avoids R.*

Actually, Theorem 1 explains better why this method allows to detect phase transitions: we shall see that knowing if $Z_c < \mathcal{P}$ holds is a crucial point with respect to the construction of one local equilibrium state which coincides with a global one. In the case $Z_c = \mathcal{P}$, there thus exists one equilibrium state which does not give positive weight to R, which means that it is not fully supported. As equilibrium state are "morally" fully supported (if they are Gibbs measure for instance), this means that there is a phase transition.

(H3) From now on, we assume that ϕ has some regularity such that for every $Z > Z_c$, $\mathcal{L}_Z$ satisfies the hypothesis of the Ionescu-Tulcea & Marinescu theorem with some Banach spaces $(\mathcal{V}, \| \|_{\mathcal{V}}) \subset \mathcal{C}^0(R)$:

(i) if $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of functions in $\mathcal{V}$ which converges in $\mathcal{C}^0(R)$ to a function φ and if for all $n \in \mathbb{N}$, $\|\varphi_n\|_{\mathcal{V}} \le C$ for some $C > 0$, then $\varphi \in \mathcal{V}$ and $\|\varphi\|_{\mathcal{V}} \le C$,
(ii) $\mathcal{L}_Z$ leaves $\mathcal{V}$ invariant and is bounded for $\| \|_{\mathcal{V}}$;
(iii) there exists $M_Z > 0$ such that $\sup_n\{\|\mathcal{L}_Z^n(\varphi)\|_\infty , \varphi \in \mathcal{V},\ \|\varphi\|_\infty \le 1\} \le M_Z < \infty$;
(iv) there exists an integer n_0 and two constants $0 < a < 1$ and $0 \le b < +\infty$ such that for all $\varphi \in \mathcal{V}$ we have $\|\mathcal{L}_Z^{n_0}(\varphi)\|_{\mathcal{V}} \le a\|\varphi\|_{\mathcal{V}} + b\|\varphi\|_\infty$;
(v) if $\mathcal{X}$ is a bounded subset of $(\mathcal{V}, \| \|_{\mathcal{V}})$ then $\mathcal{L}_Z^{n_0}(\mathcal{X})$ has compact closure in $\mathcal{C}^0(R)$.

Under these hypotheses, $\mathcal{L}_Z$ is quasi-compact on $\mathcal{V}$: the spectrum is the union of finitely many isolated complex numbers which are eigenvalues with strictly dominating modulus and the essential spectrum contained in an open disk of radius the strictly smaller than the already mentioned isolated eigenvalues. Moreover, the spectral radius λ_Z is a dominating eigenvalue (see [9]).

For instance, if ϕ is Hölder, one may consider that $\mathcal{V}$ is the set of Hölder continuous functions (with same Hölder-exponent). Actually , the spirit of the method is that in some cases, even if ϕ is less regular, one still may consider that $\mathcal{V}$ is the set of function with some Hölder regularity. This is for instance the case if up to a modification of the norm, Φ satisfies some Hölder regularity. See *e.g.* [9] or even in [12] despite the potential not being continuous[2]. Sometimes, inducing is sufficient to ensure that the induced potential Φ recovers true Hölder regularity: this is the case for the Hofbauer potential or for the Manneville-Pomeau map (see [6, 14] or Subsection 3.3).

Remark 1. We emphasize that in the case that $\mathcal{V}$ is the set of Hölder continuous functions, hypothesis (iii) of Ionescu-Tulcea & Marinescu theorem usually results from (H2). ■

Theorem 2. *For every $Z > Z_c$, there exists a unique local equilibrium state for (R, F) and $\Phi - Z.\tau(.)$. It is a Gibbs measure (with respect to F) and the pressure is $\log \lambda_Z$.*
For $Z = Z_c$, if $\mathcal{L}_{Z_c}(\mathbb{1}_R)$ converges and hypotheses of Ionescu-Tulcea & Marinescu theorem hold too, then the same result holds.

Remark 2. Construction of local equilibrium only needs the convergence of $\mathcal{L}_Z(\mathbb{1}_R)$ and that $\mathcal{L}_Z$ is quasi-compact on $\mathcal{V}$. In the case of $\mathcal{V}$ is the set of Hölder continuous functions on R, due to the form of the Hölder norm, if $\mathcal{L}_Z$ acts well on $\mathcal{V}$ for $Z > Z_c$ it also acts well for $Z = Z_c$ provided that

[2] Actually in this case the potential is Hölder continuous except on a single point where it is not continuous.

$\mathcal{L}_{Z_c}(\mathbb{1}_R)$ converges. In other words, if $\mathcal{V}$ is the set of Hölder continuous functions, local equilibrium can also be constructed for $Z = Z_c$ if and only if $\mathcal{L}_{Z_c}(\mathbb{1}_R)$ converges. ■

Assuming Theorem 2 holds, the local equilibrium state μ_Z is then of the form

$$d\mu_Z := H_Z d\nu_Z \text{ with } \mathcal{L}_Z(H_Z) = \lambda_Z H_Z \text{ and } \mathcal{L}_Z^*(\nu_Z) = \lambda_Z \nu_Z.$$

Note that λ_Z belongs to $[e^{-C}\mathcal{L}_Z(\mathbb{1}_R)(x), e^{C}\mathcal{L}_Z(\mathbb{1}_R)(x]$ for any x in R and is thus a positive real number. Continuity of H_Z and positivity of $\mathcal{L}_Z$ yield that H_Z is strictly positive. Actually, one can show that H_Z belongs to $[e^{-C}, e^{C}]$. The mixing hypothesis also shows that μ_Z has full support in R.

2.2. **Local and global equilibria.** The main question we are interested in is to know if among these μ_Z, one could find the restriction of a/the gobal equilibrium state for (X, f) and ϕ.

It is well known (see [5]) that there exists an f-invariant probability measure $\widehat{\mu}_Z$ such that $\mu_Z = \dfrac{\widehat{\mu}_Z(.)}{\widehat{\mu}_Z(.\cap R)}$ if and only if

$$\int \tau \, d\mu_Z < +\infty. \tag{2}$$

Now we have

Theorem 3. *Inequality (2) holds if and only if for some $x \in R$ $\mathcal{L}_Z(\tau)(x)$ converges.*
This is in particular the case if $Z > Z_c$.

Proof. Note that as $H_Z(x) \in [e^{-C}, e^{C}]$, $\int \tau \, d\mu_Z < +\infty$ if and only if $\int \tau \, d\nu_Z < +\infty$. Using conformality and **(H2)**, this is equivalent to convergence of $\mathcal{L}_Z(\tau)(x)$ just for (at least) one x in R. We also emphasize

$$\mathcal{L}_Z(\tau)(x) = -\frac{\partial \mathcal{L}_Z(\mathbb{1}_R)(x)}{\partial Z}.$$

This shows that $\mathcal{L}_Z(\tau)(x)$ converges if $Z > Z_c$. □

In this case the measure $\widehat{\mu}_Z$ has full support (due to mixing) and satisfies

$$h_{\widehat{\mu}_Z}(f) + \int \phi \, d\widehat{\mu}_Z = Z + \widehat{\mu}_Z(R)\log\lambda_Z. \tag{3}$$

Corollary 2.3. *For every $Z \geq \mathcal{P}$, $\lambda_Z \leq 1$.*

Proof. Equation (3) yields the desired inequality for $Z > \mathcal{P} \geq Z_c$ and continuity[3] in Z shows it also holds for $Z = Z_c$ □

One important point is that $Z \mapsto \lambda_Z$ is decreasing and analytic on $]Z_c, +\infty[$ or even on $[Z_c, +\infty[$ if $\mathcal{L}_{Z_c}(\mathbb{1}_R)$ converges. Moreover, quasi-compactness shows $\log\lambda_Z = \lim_{n\to+\infty} \frac{1}{n}\log\mathcal{L}_Z^n(\mathbb{1}_R)(x)$ for any x and Kač formula yields for $Z > Z_c$

$$\frac{d\log\lambda_Z}{dZ} = \frac{-1}{\widehat{\mu}_Z(R)}. \tag{4}$$

[3]following from the theorem of monotone convergence.

It follows that $Z \mapsto Z + \widehat{\mu}_Z(R) \log \lambda_Z$ attains its maximum either at the unique point Z where $\log \lambda_Z = 0$, or at Z_c if $\log \lambda_Z < 0$ for every $Z > Z_c$.

Then the main theorem is:

Theorem 4. *With the previous assumptions.*

- *No global equilibrium state $\widehat{\mu}$ for ϕ gives positive weight to R if,*
  - *either $\lim\limits_{Z \to Z_c} \log \lambda_Z < 0$,*
  - *or $\lim\limits_{Z \to Z_c} \log \lambda_Z = 0$ and $\lim\limits_{Z \to Z_c} \mathcal{L}(\tau) = +\infty$.*
- *If there is one global equilibrium state $\widehat{\mu}$ such that $\widehat{\mu}(R) > 0$, then, it is the unique equilibrium state with full support, if*
  - *either $\lim\limits_{Z \to Z_c} \log \lambda_Z > 0$ (and in that case $\widehat{\mu}$ is the unique global equilibrium state),*
  - *or $\lim\limits_{Z \to Z_c} \log \lambda_Z = 0$ and $\lim\limits_{Z \to Z_c} \mathcal{L}(\tau) < +\infty$.*

  Moreover, the unique Z such that $\log \lambda_Z = 0$ is $Z = \mathcal{P}$ and $\widehat{\mu} = \widehat{\mu}_{\mathcal{P}}$.

Remark 3. If (X, f) is a subshift of finite type and ϕ is continuous, Theorem 1 and existence of the global equilibrium state show that the condition $Z_c < \mathcal{P}$ yields the existence of some global equilibrium state giving positive weight to R. Consequently, there is uniqueness of the global equilibrium state and it is equal to $\widehat{\mu}_{\mathcal{P}}$. ■

3. Applications

3.1. **For non-uniformly hyperbolic dynamics.** In [12, 11] the method is used for a horseshoe with homoclinic tangency. The potential is $\phi(x) := -\log J^u(x) := -\log \det Df_{|E^u}$. It is non-continuous due to the homoclinic tangency. Authors prove the existence and uniqueness of a global equilibrium state for $\beta.\phi$ and for every $\beta \in \mathbb{R}$.

3.2. **For uniformly hyperbolic dynamics.** In [3], the method is used to produce a freezing phase transition at positive temperature with ground state supported on a quasi-crystal. Namely, it is proved that there exists a continuous potential on $\{0,1\}^{\mathbb{N}}$, say ϕ, and some $\beta_c > 0$ such that the graphs of the pressure function for $\beta.\phi$ is strictly convex for $\beta \in [0, \beta_c]$ and a half-line for $\beta \geq \beta_c$. Moreover, the exists a unique global equilibrium state for $\beta.\phi$ (may be except for $\beta = \beta_c$). It has full support for $\beta < \beta_c$ and is supported on some uniquely ergodic and zero-entropy (and different to a periodic orbit) for $\beta > \beta_c$.

In [8] the method is used to construct a mixing system with an non-flat phase transition. It is also proved that after the transition, the system may have co-existence of several global equilibrium states despite the pressure function remains analytic.

In [10], the method is used to prove convergence at temperature zero of the global equilibrium state in a subshift of finite type and for a locally constant potential. Inducing scheme allows to control the different basins of different ergodic ground states, and to estimate how their relative weights vary in function of the inverse of the temperature β.

3.3. **Hofbauer potential or Manneville-Pomeau map.** To finish with a simple example, we apply the method to the Hofbauer potential in $\{0,1\}^{\mathbb{N}}$

$$\phi(x) = \begin{cases} -\log(1 + \frac{1}{n}) \text{ if } x = 0^n 1 \ldots, \\ -A < 0 \text{ if } x = 1 \ldots. \end{cases}$$

This case is usually associated to the Manneville-Pomeau map, say *e.g.*

$$f:[0,1]\circlearrowleft\ x\mapsto\begin{cases}\dfrac{x}{1-x}\text{ if } x\in[0,\dfrac12],\\ 2x \mod 1 \text{ if } x\in[\frac12,1].\end{cases}$$

One can find in [1] a description of why these two cases are associated, and actually similar.

In that case we induce on the cylinder [1]. Note that only one orbit does not enter into [1], and it is $0^\infty=000\ldots$. Moreover, for any $x\in[1]$, and for every $\beta>0$

$$\mathcal{L}_{\beta,Z}(\mathbb{1}_{[1]})(x)=e^{-\beta.A}\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\right)^{\beta}e^{-Z.(n+1)}.$$

For every $\beta\ge0$, this series converges if $Z>0$ and diverges for $Z<0$. Therefore $Z_c=0$, and we point-out that

$$0=h_{\delta_{0^\infty}}+\beta.\int\phi\,d\delta_{0^\infty}.$$

which is the *ad'hoc* reformulation of Theorem 1.

Now, the form of the potential also yields $\lambda_{\beta,Z}=\mathcal{L}_{\beta,Z}(\mathbb{1}_{[1]})(x)$ for any x in [1]. Let us study the critical case $Z=Z_c$:

$$\lambda_{\beta,0}:=e^{-\beta.A}\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\right)^{\beta}.$$

For $\beta\le1$, $\lambda_{\beta,0}=+\infty$. Furthermore, the function $\beta\mapsto\lambda_{\beta,0}$ is decreasing on $]1,+\infty[$, goes to $+\infty$ if $\beta\to1$ and goes to 0 if $\beta\to+\infty$. Therefore, there exists a unique β_c such that $\lambda_{\beta_c,0}=1$.

For $\beta>\beta_c$, no equilibrium state gives positive weight to [1], which means that δ_{0^∞} is the unique equilibrium state and the pressure is 0.

For $\beta<\beta_c$, the map $Z\mapsto\lambda_{\beta,Z}$ is decreasing, and there is a unique $Z=\mathcal{P}(\beta)>0$ such that

$$\lambda_{\beta,\mathcal{P}(\beta)}=1.$$

As $\mathcal{P}(\beta)>0=Z_c$, we are in the case of Theorem 3, and the associated measure $\widehat{\mu}_{\mathcal{P}(\beta)}$ satisfies

$$h_{\widehat{\mu}_{\mathcal{P}(\beta)}}(\sigma)+\beta\int\phi\,d\widehat{\mu}_{\mathcal{P}(\beta)}=\mathcal{P}(\beta)>0.$$

This last inequality shows that δ_{0^∞} cannot be an equilibrium state, hence, there exists an equilibrium state which gives positive weight to [1], and it is $\widehat{\mu}_{\mathcal{P}(\beta)}$.

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